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LETTER TO THE EDITOR

On the identification of quasiprimary scaling operators in local scale-invariance

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Abstract
The relationship between physical observables defined in lattice models and the associated (quasi-)primary scaling operators of the underlying field theory is revisited. In the context of local scale-invariance, we argue that this relationship is only defined up to a time-dependent amplitude and derive the corresponding generalizations of predictions for two-time response and correlation functions. Applications to non-equilibrium critical dynamics of several systems, with a fully disordered initial state and vanishing initial magnetization, including the Glauber–Ising model, the Frederikson–Andersen model and the Ising spin glass are discussed. The critical contact process and the parity-conserving non-equilibrium kinetic Ising model are also considered.

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(Some figures in this article are in colour only in the electronic version)

The analysis of the collective behaviour of many-body systems is greatly helped in situations where some scale invariance allows an efficient description through field-theoretical methods. A necessary requirement for the application of these is the possibility of identifying the physical observables typically defined in terms of a lattice model, e.g. $\sigma_{\boldsymbol{r}}$ for the order-parameter at the site $\boldsymbol{r}$, with a continuum field $\phi(\boldsymbol{r})$ (called a *scaling operator* [1]) with well-defined scaling

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properties $\phi(\boldsymbol{r}) = \mathfrak{b}^{-x}\phi(\boldsymbol{r}/\mathfrak{b})$. In other words, one generally expects that the correspondence ($\mathfrak{a}$ is the lattice constant)

$$\sigma_{\boldsymbol{r}} \rightarrow \mathfrak{a}^{-x}\phi(\boldsymbol{r}) \tag{1}$$

can be defined in equilibrium systems or more generally in steady states of non-equilibrium systems, see e.g. [1–3]. In addition, in equilibrium systems one expects the same sort of relationship to hold true where $\phi(\boldsymbol{r})$ is now a primary scaling operator of a conformal field theory and allows space-dependent rescaling factors $\mathfrak{b} = \mathfrak{b}(\boldsymbol{r})$ [1].

In this letter, we reconsider this correspondence for systems with dynamical scaling and far from equilibrium, as it occurs for example in ageing phenomena. Concrete examples are phase-ordering kinetics or non-equilibrium critical dynamics, see [4–6] for reviews. Among the main quantities of interest are the two-time autocorrelation function $C(t,s)$ and the autoresponse function $R(t,s)$:

$$\begin{aligned} C(t,s) &= \langle \phi(t,\boldsymbol{r})\phi(s,\boldsymbol{r})\rangle = s^{-b} f_C(t/s) \\ R(t,s) &= \left.\frac{\delta\langle\phi(t,\boldsymbol{r})\rangle}{\delta h(s,\boldsymbol{r})}\right|_{h=0} = \langle \phi(t,\boldsymbol{r})\widetilde{\phi}(s,\boldsymbol{r})\rangle = s^{-1-a} f_R(t/s), \end{aligned} \tag{2}$$

where $\widetilde{\phi}$ is the response field in the Janssen-de Dominicis formalism [7, 8], a and b are the ageing exponents and f_C and f_R are the scaling functions such that $f_{C,R}(y) \sim y^{-\lambda_{C,R}/z}$ for $y \gg 1$. These scaling forms are only valid in the scaling regime where $t, s \rightarrow \infty$ *and* $y = t/s > 1$ fixed. Until recently, the scaling (2) has only been studied for systems with a fully disordered initial state with mean initial magnetization $m_0 = \langle\phi(0,\boldsymbol{r})\rangle = 0$. The study of the effects of a non-vanishing initial magnetization on the ageing behaviour is only beginning [9–11]. We stress that in the kind of system under consideration invariance under time-translations is broken. In an attempt to try to derive the form of the scaling functions in a model-independent way it has been argued [12] that the scaling operators ϕ and $\widetilde{\phi}$ should transform covariantly under a larger group than mere dynamical scale-transformations. If such an invariance exists, one may call it a *local scale-invariance (LSI)*[9]. The infinitesimal generators of local scale-invariance read [12–14]

$$X_0 = -t\partial_t - \frac{x}{z}, \qquad X_1 = -t^2\partial_t - \frac{2}{z}(x+\xi)\,t, \tag{3}$$

where for simplicity we have suppressed the terms acting on the space coordinates which are not important for what follows. We have also not written down the further generators of LSI which do not modify the time t but only act on the space coordinates $\boldsymbol{r}$, and the absence of any scaling of m_0 means that we are restricting ourselves to the case $m_0 = 0$ throughout. Here, x is the scaling dimension of the scaling operator $\phi(t,\boldsymbol{r}) = \mathfrak{b}^{-x/z}\phi(t/\mathfrak{b}^z, \boldsymbol{r}/\mathfrak{b})$, where z is the dynamical exponent and ξ is a constant. It is the purpose of this letter to clarify the meaning of this constant ξ.

Motivated by the analogy with two-dimensional conformal invariance, we generalize the dilatation generator X_0 and the generator X_1 of 'special' transformations as follows to all $n \geqslant 0$:

$$X_n = -t^{n+1}\partial_t - \frac{x}{z}(n+1)t^n - \frac{2\xi}{z}nt^n \tag{4}$$

[9] All existing tests of LSI have been performed for $m_0 = 0$.

such that the commutator $[X_n, X_m] = (n-m)X_{n+m}$ holds for all $n, m \in \mathbb{N}_0$ (with the convention $0 \in \mathbb{N}_0$).[10] Next, the global form of these transformations reads as follows. If $t = \beta(t')$ such that $\beta(0) = 0$, then $\phi(t)$ transforms as

$$\phi(t) = \dot{\beta}(t')^{-x/z} \left(\frac{t'\dot{\beta}(t')}{\beta(t')} \right)^{-2\xi/z} \phi'(t') \tag{5}$$

where again the space dependence of ϕ was suppressed. The infinitesimal generators X_n are recovered for $\beta(t) = t + \epsilon t^{n+1}$, with $|\epsilon| \ll 1$. From this, it is clear that ϕ is not transforming as an usual primary scaling operator. But if one defines $\Phi(t) := t^{-2\xi/z}\phi(t)$ the scaling operator $\Phi(t)$ becomes a conventional primary scaling operator of LSI, namely

$$\Phi(t) = \dot{\beta}(t')^{-(x+2\xi)/z} \Phi'(t') \tag{6}$$

but with a modified scaling dimension $x \to x + 2\xi$. In other words, if time-dependent observables of lattice models $\sigma_{\mathbf{r}}(t)$ can be related to a primary scaling operator $\Phi(t)$ at all, it should be via the relation

$$\sigma_{\mathbf{r}}(t) \to \mathfrak{a}^{-x}\phi(t) = \mathfrak{a}^{-x}\, t^{2\xi/z}\Phi(t) \tag{7}$$

rather than by equation (1). Of course, (7) is only possible because of the absence of time-translation invariance. We emphasize that the scaling of ϕ is unusual in that under a dilatation $t \to \mathfrak{b}^z t$ the scaling dimension remains x but for more general scale transformations a new effective scaling dimension $x + 2\xi$ appears.

As a simple application, consider the two-time autoresponse function. For quasi-primary scaling operators $\Phi(t)$ and $\widetilde{\Phi}(s)$ with scaling dimensions x and $\widetilde{x}$, respectively, local scale-invariance with $m_0 = 0$ predicts $\langle \Phi(t)\widetilde{\Phi}(s)\rangle = (t/s)^{(\widetilde{x}-x)/z}(t-s)^{-(x+\widetilde{x})/z}$, up to normalization, as shown in [12]. In view of (7), the physical autoresponse function rather reads

$$\begin{aligned} R(t,s) = \langle \phi(t)\widetilde{\phi}(s)\rangle &= \langle t^{2\xi/z}\Phi(t) s^{2\widetilde{\xi}/z}\widetilde{\Phi}(s)\rangle \\ &= s^{-(x+\widetilde{x})/z} \left(\frac{t}{s}\right)^{(2\widetilde{\xi}+\widetilde{x}-x)/z} \left(\frac{t}{s}-1\right)^{-(x+\widetilde{x}+2\xi+2\widetilde{\xi})/z} \\ &= s^{-1-a} \left(\frac{t}{s}\right)^{1+a'-\lambda_R/z} \left(\frac{t}{s}-1\right)^{-1-a'} \end{aligned} \tag{8}$$

(up to normalization) and the effective scaling dimensions of $\Phi(t)$ and $\widetilde{\Phi}(s)$ are read off from equation (6) to be now $x + 2\xi$ and $\widetilde{x} + 2\widetilde{\xi}$, respectively. In the last line, we have re-introduced the standard exponents a, a' and λ_R and hence reproduce the result quoted in [14]. Early discussions of local scale-invariance had assumed $a' = a$ from the outset. In the appendix, we discuss the scaling form of the autocorrelator $C(t,s)$ in those cases where $z = 2$.

It appears that the more general correspondence (7) and consequently the response (8) with $a' \neq a$ actually occurs in non-equilibrium critical dynamics, as we shall now illustrate in a few examples. We stress that in the models considered here (with the only exception of the contact process) we always use a fully disordered initial state with a vanishing initial magnetization $m_0 = 0$.[11] In table 1 we collect results on the exponents a, a' and λ_R/z in some

[10] This is the unique semi-infinite extension of the algebra $\langle X_0, X_1\rangle$ which does not introduce further differential operators into X_n and is compatible with equation (3).

[11] From LSI, it is then easy to see that the time-dependent magnetization $m(t) = m_0 = 0$ for all times, in agreement with the Monte Carlo and the exact results. On the other hand, if initially $m_0 > 0$, one has the regime of short-time dynamics with $m(t) \sim t^{\theta}$ [15] before the long-time decay $m(t) \sim t^{-\beta/(\nu z)}$ [16]. The scaling of two-time observables has been recently discussed in [9–11] and it was shown that for $m_0 \neq 0$ the universal scaling behaviour is different from the one found for $m_0 = 0$. An extension of LSI to non-equilibrium critical dynamics with non-vanishing initial magnetizations is an open problem to which we hope to return elsewhere.

Table 1. Values of the exponents a, a' and λ_R/z in several non-disordered and a few glassy systems which are at a critical point of their stationary state. If a numerical result is quoted without an error bar it is taken form the literature, otherwise the numbers in brackets give our estimate of the uncertainty in the last digit(s). NEKIM stands for non-equilibrium kinetic Ising model with conserved parity and FA stands for the Frederikson–Andersen model. The methods of calculation of the two-time autoresponse are D: direct space, P: momentum space, A: alternating external field; E refers to an exact solution and N to a numerical study.

Model	a	$a' - a$	λ_R/z	Method	Reference
OJK model	$(d-1)/2$	$-1/2$	$d/4$	D,E	[14, 17, 18]
1D Ising	0	$-1/2$	$1/2$	D,E	[13, 19]
2D Ising	0.115	−0.187(20)	0.732(5)	P,N	[22]
3D Ising	0.506	−0.022(5)	1.36	P,N	[22]
1D contact process	−0.681	+0.270(10)	1.76(5)	D,N	[26, 27]
1D NEKIM	−0.430	−0.09	0.56	D,N	[28]
FA, $d > 2$	$1 + d/2$	−2	$2 + d/2$	P,E	[20]
FA, $d = 1$	1	$-3/2$	2	P,E	[20, 21]
3D Ising spin glass	0.060(4)	−0.76(3)	0.38(2)	A,N	[14]

models with a critical stationary state and where $a' \neq a$.[12] In several cases, these exponents can be read off from the exact solution, i.e., for the magnetic response in the OJK model [17, 18] and the 1D Glauber–Ising model at zero temperature [19] or else the energy response in the zero-temperature Frederikson–Andersen model [20, 21].

Another interesting test case is provided by the critical Ising model in 2D and 3D. Indeed, it was pointed out some time ago that the numerical calculation of the two-time response function $\widehat{R}_{\boldsymbol{q}}(t,s) = \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{r}\, R(t,s;\boldsymbol{r})\, \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}$ in *momentum* space provides a more sensitive test on the form of its scaling function than in direct space [22]. The methods of LSI can be readily adapted to momentum space and the analogue of (8) is, again up to normalization and for $m_0 = 0$,

$$\widehat{R}_{\mathbf{0}}(t,s) = s^{-1-a+d/z} \left(\frac{t}{s}\right)^{1+a'-\lambda_R/z} \left(\frac{t}{s} - 1\right)^{-1-a'+d/z}. \tag{9}$$

Since measurements of autoresponse functions are much affected by statistical noise, one often rather studies integrated response functions. Here we consider

$$\chi_{\text{Int}}(t,s) := \int_{s/2}^{s} \mathrm{d}u\, \widehat{R}_{\mathbf{0}}(t,u) = \chi_0 s^{-a+d/z} f_\chi(t/s) \tag{10}$$

which is free from effects that mask the true scaling behaviour in several other variants of integrated responses [22]. The scaling function $f_\chi(y)$ follows from LSI, equation (9):

$$\begin{aligned} f_\chi(y) = y^{(d-\lambda_R)/z} \Bigg[{}_2F_1\left(1 + a' - \frac{d}{z}, \frac{\lambda_R}{z} - a; 1 + \frac{\lambda_R}{z} - a; \frac{1}{y}\right) \\ - 2^{a-\lambda_R/z}\, {}_2F_1\left(1 + a' - \frac{d}{z}, \frac{\lambda_R}{z} - a; 1 + \frac{\lambda_R}{z} - a; \frac{1}{2y}\right)\Bigg] \end{aligned} \tag{11}$$

and where ${}_2F_1$ is Gauss' hypergeometric function. In figure 1 we compare simulational data [22] with this prediction for both the 2D and 3D critical Ising model with non-conserved heat-bath dynamics. It had already been observed before [22] that local scale-invariance with the additional assumption $a' = a$ does not agree with the numerical data in 2D and only

[12] In table 1, D,E means that the exact response agrees with (8) with the given values of the exponents, while P,E means that there is exact agreement with (9).

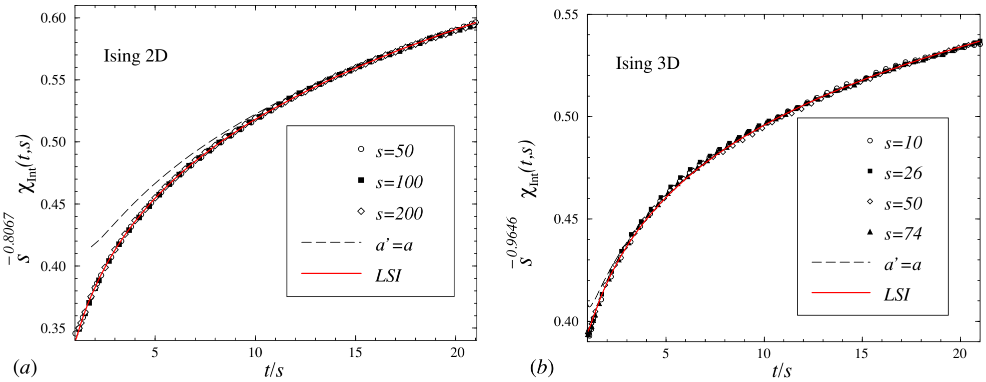

Figure 1. Intermediate susceptibility $\chi_{\rm Int}(t, s)$ in momentum space in the (*a*) 2D and (*b*) 3D critical Ising model, for several values of the waiting time s. The full curve is the LSI prediction equations (10), (11) with the exponents as listed in table 1. The dashed line corresponds to the case $a' = a$.

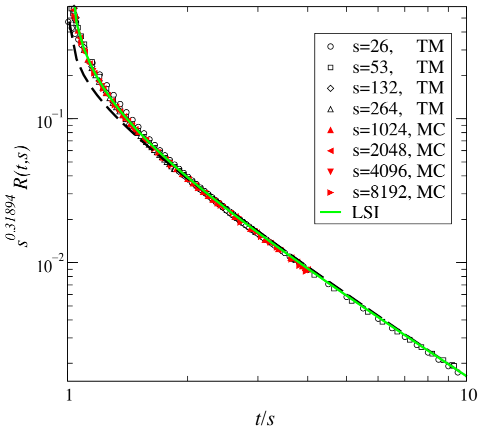

Figure 2. Autoresponse function for the critical 1D contact process for several waiting times s. The data labelled TM come from the transfer matrix renormalization group [26] and MC denotes Hinrichsen's Monte Carlo data [27]. The dashed line corresponds to the case $a' = a$ and the full curve gives the LSI prediction equation (8) with the exponents as listed in table 1.

marginally so in 3D and we confirm this finding. However, we also see that the data can be perfectly matched by LSI, within the numerical precision, if a and a' are allowed to be different. We did check that the integrated TRM response functions in direct space as studied in [23] do not change appreciably with $a' - a$.

A similar conclusion can also be drawn for the 1D critical contact process. It has been shown recently that the phenomenology of ageing can also be found in critical stochastic processes although these do *not* satisfy detailed balance and have a non-equilibrium steady state [24–26]. In figure 2 we compare the numerical data obtained directly for $R(t, s)$ either from the LCTMRG [26] or Monte Carlo simulations [27]. It is satisfying that the data from both methods are consistent with each other in the scaling regime, where s and $t - s$ are both large enough. Again, we observe an almost perfect agreement with equation (8), provided $a' \neq a$.[13]

On the other hand, when one looks closer at the region where $t/s \lessapprox 1.1$, one does observe deviations of the data from (8) [27]. In trying to analyse this, recall that non-equilibrium *critical* dynamics is special in the sense that both the ageing regime (where $t - s \sim \mathrm{O}(s)$) and the quasistationary regime (where $t - s \ll s$) display dynamical scaling with the same length scale $L(t) \sim t^{1/z}$, where z is the equilibrium dynamical critical exponent. Hence, one usually expects some 'crossover' to occur. In terms of the response function, this might be formalized by writing $R = \mathcal{R}(s/\tau_*, (t-s)/\tau_*, s)$ where τ_* is some reference time scale such that, with $(t-s)/\tau_* = \mathrm{O}(1)$,

$$\lim_{s\to\infty} R = \begin{cases} R_{\mathrm{eq}}(t-s) & \text{for} \quad s/\tau_* \to \infty \\ s^{-1-a} f_R(t/s) & \text{for} \quad s/\tau_* = \mathrm{O}(1). \end{cases}$$

Since in lattice calculations s is always finite, the 'crossover' can be illustrated by studying $Q := R(t,s)/R_{\mathrm{eq}}(t-s) \sim R(t,s)(t-s)^{1+a}$. As long as LSI still describes the data, one expects $Q \sim (y-1)^{a-a'}$ for $y = t/s \gtrapprox 1$ and deviations from it should signal the presumed crossover to the quasistationary regime $y \to 1$ where $Q(y)$ should become constant. For the 1D critical contact process we find that for $y = t/s \lessapprox 1.1$, $Q(y)$ still obeys scaling for s large enough and that Q changes from $Q \approx 0.3$ at $y - 1 \approx 0.1$ to $Q \approx 0.8$ at $y - 1 \approx 10^{-4}$. $Q(y)$ appears to become flatter as $y \to 1$, but the change to a quasistationary behaviour could not yet be observed, in spite of large waiting times $s > 860\,000$, before strong finite-time effects set in at $t - s = \mathrm{O}(1)$. In comparison, unpublished data for the 2D Ising model [27] show convergence towards $Q(y) \sim (y-1)^{0.187}$ as s increases before finite-time effects destroy scaling. We conclude that LSI does accurately describe the data as long as t/s is large enough such that the effects of the 'crossover' are not yet noticeable. A quantitative analysis of data from the region $t/s \lessapprox 1.1$ would require a precise theory of the 'crossover' between the ageing regime and the region $t - s \ll s$, the rôle of finite-time effects and the influence of initial conditions, e.g. different initial fillings of the lattice.

Very recently, a similar test was carried out in a 1D kinetic Ising model with competing Glauber and Kawasaki dynamics [28]. The stationary state is therefore not an equilibrium state. This was the first time that LSI was tested and confirmed for a dynamics where the parity of the total spin is conserved by the dynamics.

Finally, we recall that studying the scaling behaviour of an alternating susceptibility gives yet another direct access to the exponent $a' - a$. This was applied to the critical 3D Ising spin glass [14], with a binary distribution of the couplings $J_{i,j} = \pm J$.

In summary, we have reconsidered the way how observables defined in non-equilibrium lattice models might be related to (quasi-)primary scaling operators of field theory. Our result equation (7) points to a so far overlooked subtlety which might be of relevance in the discussion of the functional form of non-equilibrium scaling functions, for example in ageing phenomena. It remains to be seen how general the phenomenon for which we have presented evidence really is[14]. The results on $R(t,s)$ as collected in table 1 for the non-equilibrium dynamics of some models with $m_0 = 0$ appear to be compatible with the predictions equations (8), (9) of local scale-invariance, provided crossover effects to non-ageing regimes are negligible. The

[13] Hinrichsen quotes $\lambda_R/z \approx 1.75$ and $1 + a' \approx 0.59$ [27] in good agreement with our estimates. The contact process is the only known example where $a' - a > 0$.

[14] It is not inconceivable that analogues might exist in equilibrium critical phenomena, for instance when spatial translation-invariance is broken by disorder or boundaries.

multitude of examples in table 1 suggests that rather being a kind of exotic exception (a belief implicit in [12–14]), the case $a' \neq a$ might turn out to be the generic situation. Having seen that the same mechanism also explains the exact autocorrelator of the 1D Glauber–Ising model indicates that the correspondence (7) should be more than just a patching-up of data for the autoresponse function.

What does this mean for the existence of local scale-invariance in non-equilibrium dynamics, with $m_0 = 0$? In a few exactly solved systems (where the dynamical exponent $z = 2$) we have found exact agreement and in several models as generic as kinetic Ising models or the contact process equations (8), (9) describe the data very well for t/s not too small. It is remarkable that the two-time autocorrelations and autoresponses of models as physically different as those included in table 1 (and several further ones with $a = a'$ which we did not include) can be described in terms of a single theoretical idea. On the other hand, field-theoretical studies of the critical O(n) model in both $4 - \varepsilon$ dimensions [5, 29] and in $2 + \varepsilon$ dimensions [11], although they agree with LSI at the lowest orders in ε, continue to find discrepancies with either (8) or (9) at some higher order. However, non-equilibrium field theory presently only yields explicit results for the first few terms of the ε-expansion series. When one truncates this series to an ε-dependent sum, the resulting numerical values for the scaling functions are still far from the numerical data[15]. But since we have shown that LSI reproduces the known exact results of both $R(t, s)$ and $C(t, s)$ of the 1D Ising model it might be too simplistic to argue that LSI could at best describe Gaussian fluctuations. A better understanding of the dynamical symmetries of non-equilibrium critical dynamics remains a challenging problem.

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Appendix. Two-time autocorrelations for $z = 2$

If the dynamical exponent $z = 2$, local scale-invariance reduces to Schrödinger invariance. We have already described in the past [13] how two-time autocorrelation functions can be calculated in the case $\xi = 0$ and we now wish to extend that treatment to the more general correspondence (7). We consider a Langevin equation of the form $\partial_t \phi = -D \frac{\delta \mathcal{H}}{\delta \phi} - D v(t) \phi + \eta$, where $\mathcal{H}$ is the Hamiltonian, D the diffusion constant, the Gaussian noise η has zero mean and variance $\langle \eta(t, \boldsymbol{r}) \eta(s, \boldsymbol{r}') \rangle = 2DT\delta(t - s)\delta(\boldsymbol{r} - \boldsymbol{r}')$ and T is the bath temperature. The potential $v(t)$ acts as a Langrange multiplier which can be used to describe explicitly the breaking of time-translational invariance. Here we restrict to situations where

$$k(t) := \exp\left[-D \int_0^t \mathrm{d}u\, v(u)\right] \sim t^F. \tag{A.1}$$

[15] The second-order calculation in $4 - \varepsilon$ dimensions for $n = 1$ is a little closer to the numerical data than LSI with $a' = a$ [22].

Then is has been shown [13] that for systems at criticality

$$C(t,s) = \langle \phi(t)\phi(s)\rangle = DT_c \int \mathrm{d}u\, \mathrm{d}\boldsymbol{R}\, \langle \phi(t,\boldsymbol{y})\phi(s,\boldsymbol{y})\widetilde{\phi}^2(u,\boldsymbol{R}+\boldsymbol{y})\rangle_0$$
$$= DT_c \int \mathrm{d}u\, \mathrm{d}\boldsymbol{R}\, \frac{k(t)k(s)}{k(u)^2} \mathcal{R}_0^{(3)}(t,s,u;\boldsymbol{R}), \tag{A.2}$$

where $\mathcal{R}_0^{(3)}$ is the well-known three-point response function for $v(t) = 0$ which can be found from its Schrödinger covariance and reads [30]

$$\mathcal{R}_0^{(3)}(t,s,u;\boldsymbol{r}) = \mathcal{R}_0^{(3)}(t,s,u) \exp\left[-\frac{\mathcal{M}}{2}\frac{t+s-2u}{(s-u)(t-u)}\boldsymbol{r}^2\right] \Psi\left(\frac{t-s}{(t-u)(s-u)}\boldsymbol{r}^2\right)$$

$$\mathcal{R}_0^{(3)}(t,s,u) = \Theta(t-u)\Theta(s-u)(t-u)^{-\widetilde{x}_2}(s-u)^{-\widetilde{x}_2}(t-s)^{-x+\widetilde{x}_2},$$

where Ψ is an undetermined scaling function and the causality conditions $t > u, s > u$ are noted explicitly. In writing this, we have dropped a term coming from the correlations in the initial state which merely produces finite-time corrections to the leading scaling behaviour, see [4, 5, 13].

We now generalize this to the primary scaling operators according to (7). The operator Φ has the scaling dimension $x + 2\xi$ and the composite scaling operator $\widetilde{\Phi}^2$ has the scaling dimension $2\widetilde{x}_2 + 4\widetilde{\xi}_2$.[16] We then obtain for the physical autocorrelation function, up to normalization and with $t > s$

$$C(t,s) = (ts)^{\xi} \int \mathrm{d}u\, \mathrm{d}\boldsymbol{R} \langle \Phi(t,\boldsymbol{y})\Phi(s,\boldsymbol{y})\widetilde{\Phi}^2(u,\boldsymbol{R}+\boldsymbol{y})\rangle_0\, u^{2\widetilde{\xi}_2}$$
$$= (ts)^{\xi}(t-s)^{-x-2\xi+\widetilde{x}_2+2\widetilde{\xi}_2-d/2} \int_0^s \mathrm{d}u \frac{k(t)k(s)}{k(u)^2} u^{2\widetilde{\xi}_2}\, [(t-u)(s-u)]^{-\widetilde{x}_2-2\widetilde{\xi}_2+d/2}$$
$$\times \int \mathrm{d}\boldsymbol{R}\, \exp\left[-\frac{\mathcal{M}}{2}\frac{t+s-2u}{t-s}\boldsymbol{R}^2\right] \Psi(\boldsymbol{R}^2)$$
$$= s^{1+d/2-x-\widetilde{x}_2} \left(\frac{t}{s}\right)^{\xi+F} \left(\frac{t}{s}-1\right)^{\widetilde{x}_2+2\widetilde{\xi}_2-x-2\xi-d/2}$$
$$\times \int_0^1 \mathrm{d}v\, v^{2\widetilde{\xi}_2-F} \left[\left(\frac{t}{s}-v\right)(1-v)\right]^{d/2-\widetilde{x}_2-2\widetilde{\xi}_2} \boldsymbol{\Psi}\left(\frac{t/s+1-2v}{t/s-1}\right) \tag{A.3}$$

and where the function $\boldsymbol{\Psi}$ is defined by the integral over $\boldsymbol{R}$. By comparison with the standard scaling form for $C(t,s)$, we read off $b = x + \widetilde{x}_2 - 1 - d/2$ and $\lambda_C = 2(x - F) + 2\xi$.[17] Furthermore, since $1 + a' = x + 2\xi$, it turns out that the form of the scaling function $f_C(y)$ is described by just one more parameter $\mu := \xi + \widetilde{\xi}_2$ and we finally have

$$C(t,s) = C_0 s^b \left(\frac{t}{s}\right)^{1+a'-\lambda_C/2} \left(\frac{t}{s}-1\right)^{b-2a'-1+2\mu}$$
$$\times \int_0^1 \mathrm{d}v\, v^{\lambda_C+2\mu-2-2a'} \left[\left(\frac{t}{s}-v\right)(1-v)\right]^{a'-b-2\mu} \boldsymbol{\Psi}\left(\frac{t/s+1-2v}{t/s-1}\right) \tag{A.4}$$

[16] For bosonic free fields, one would have $\widetilde{x}_2 = \widetilde{x}$ and $\widetilde{\xi}_2 = \widetilde{\xi}$.

[17] A similar calculation for the autoresponse function gives, up to normalization,

$$R(t,s) = s^{-(x+\widetilde{x})/2}(t/s)^{\xi+F}\,(t/s-1)^{-x-2\xi}\,\delta_{x+2\xi,\widetilde{x}+2\widetilde{\xi}}$$

which reproduces again equation (8), hence $\lambda_R = 2(x - F) + 2\xi = \lambda_C$ as expected [4] for non-disordered systems without long-range initial correlations. In particular, for critical systems with $a = b$ the equality $\lambda_C = \lambda_R$ implies that there is a finite limit fluctuation–dissipation ratio $X_\infty = \lim_{(t/s)\to\infty} R(t,s)/(T_c \partial C(t,s)/\partial s)$, see [5].

and we have also reintroduced a normalization constant C_0. This should hold for simple (non-glassy) magnets with $z = 2$ and in situations where the initial correlations have no effect on the leading scaling behaviour; of course the scaling limit $s \to \infty$ and $t/s = y > 1$ fixed is understood.

As an illustration, we consider the 1D Glauber–Ising model. At $T = 0$, the exact two-time autocorrelation function is [19]

$$C(t,s) = \frac{2}{\pi}\arctan\left(\sqrt{\frac{2}{t/s-1}}\right). \tag{A.5}$$

This holds true not only for the usually considered short-ranged initial conditions but also for long-ranged initial spin–spin correlations $\langle\sigma_r(0)\sigma_0(0)\rangle \sim r^{-\nu}$ with $\nu > 0$ (for $\nu = 0$ an analogous result holds for the connected autocorrelator) [19]. In addition, the exponents a, a' and λ_R are independent of ν.

In previous work [13], we have already explained the form of the exact autoresponse function $R(t,s)$ in terms of the correspondence equation (7) (see table 1) but we had to leave open the analogous question for $C(t,s)$. In order to account for (A.5), we remark that for $t = s$, the autocorrelator should not be singular. This requires

$$\Psi(w) = w^{b-2a'-1+2\mu} \qquad \text{for} \quad w \gg 1. \tag{A.6}$$

The most simple way to realize this is to require that (A.6) holds for all values of w. This kind of assumption was already seen to become exact in models described by an underlying bosonic free field-theory [13]. Recalling from table 1 that $b = a = 0$ and $\lambda_C = 1$ and assuming (A.6) to hold for all w, we obtain

$$C(t,s) \approx C_0 \int_0^1 \mathrm{d}v\, v^{2\mu}\left[\left(\frac{t}{s}-v\right)(1-v)\right]^{-2\mu-1/2}\left(\frac{t}{s}+1-2v\right)^{2\mu}. \tag{A.7}$$

Because the exact result (A.5) is independent of the initial correlations, the comparison with the expression (A.7) derived from the thermal noise is justified. The exact Glauber–Ising result (A.5) is recovered from (A.7) for $\mu = -1/4$ and $C_0 = \sqrt{2}/\pi$.

This is the first example of an exactly solved model with $a' \neq a$ where the scaling of *both* the autoresponse and of the autocorrelation functions can be explained in terms of LSI.

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